Auditorium Exercise 02

Differential Equations II for Students of Engineering Sciences
Summer Semester 2024

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Announcements

Exercise class Monday, May 6th

(Very likely) no Exercise class here, may be online or attend German ones (more info in StudIP)

Lectures

There will be no (English) recordings if no one attends the lectures

Overview

Classification of partial differential equations

Linearity of partial differential equations

Superposition

Linear substitution

Classification of partial differential equations

1. **Scalar PDEs** (compare lecture p.4-6)

   $$F(x, u(x), Du(x), D^2u(x), \dots, D^m u(x)) = 0 \qquad \forall x \in \Omega \subset \mathbb{R}^n$$

   m : order (=highest derivative order)

   $n \geq 2$: dimension (space where x lives)
2. **Linear PDEs**: Affine linear in $u, Du, D^2u, \dots, D^m u$ with coefficients depending on x.
3. **Semilinear PDEs**: Affine linear (only) in $D^m u$ with coefficients depending on x.
4. **Quasilinear PDEs**: Affine linear (only) in $D^m u$ ab with coefficients depending on x and $u, Du, \dots, D^{m-1}u$.
5. **Fully non linear PDEs:** Not quasilinear.

Examples

1. $au_x + bu_y = g(x, y) \qquad g$ cont., $a, b \in \mathbb{R}$, $a \cdot b \neq 0$.

   order: coefficients:
   inhomogenity: classification:

2. $x^2 u_t + tu_x = 0$

   order: coefficients:
   inhomogenity: classification:

3. Telegrapher's equations: $u_{tt} - u_{xx} + 2u_t + u = 0$

   order: coefficients:
   inhomogenity: classification:

4. $u_t + (u^4)_x = 0$

   order: coefficients:
   inhomogenity: classification:

5. $(u_{xx})^2 + u_{xy} + 2u_y = 0$

   order: coefficients:
   inhomogenity: classification:

Examples

1. $au_x + bu_y = g(x,y)$ $\quad$ g cont., $a, b \in \mathbb{R}$, $a \cdot b \neq 0$.

   order: 1 $\quad$ coefficients: a,b
   inhomogenity: $g(x,y)$ $\quad$ classification: linear

2. $x^2 u_t + t u_x = 0$

   order: 1 $\quad$ coefficients: x^2, t
   inhomogenity: 0 $\quad$ classification: linear

3. Telegrapher's equations: $u_{tt} - u_{xx} + 2u_t + u = 0$

   order: 2 $\quad$ coefficients: 1,-1
   inhomogenity: 0 $\quad$ classification: linear

4. $u_t + (u^4)_x = 0$

   order: 1 $\quad$ coefficients: $4u^3$
   inhomogenity: 0 $\quad$ classification: quasilinear

5. $(u_{xx})^2 + u_{xy} + 2u_y = 0$

   order: 2 $\quad$ coefficients: -
   inhomogenity: 0 $\quad$ classification: Fully non linear

Linearity of PDEs: Examples

Check whether multiples of solutions of the PDE

$$u_t + (u^4)_x = 0$$

are also solutions. If yes, check whether sums of solutions are also solutions.

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- Assume $\tilde{u}_t + (\tilde{u}^4)_x = 0$

  Thus, $\tilde{u}_t + 4\tilde{u}^3 \cdot \tilde{u}_x = 0 \Longrightarrow$

  For $\hat{u} := k \cdot \tilde{u}$ with $k \in \mathbb{R}$, $k \neq 0$ holds

$$\hat{u}_t + (\hat{u}^4)_x = (k \cdot \tilde{u})_t + 4(k\tilde{u})^3(k\tilde{u})_x =$$

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Check whether multiples of solutions of the PDE

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- Assume $\tilde{u}_t + (\tilde{u}^4)_x = 0$

  Thus, $\tilde{u}_t + 4\tilde{u}^3 \cdot \tilde{u}_x = 0 \implies \tilde{u}_t = -4(\tilde{u})^3 \tilde{u}_x$

  For $\hat{u} := k \cdot \tilde{u}$ with $k \in \mathbb{R}$, $k \neq 0$ holds

$$\hat{u}_t + (\hat{u}^4)_x = (k \cdot \tilde{u})_t + 4(k\tilde{u})^3(k\tilde{u})_x = \underbrace{(k^4 - k)\tilde{u}^3 \tilde{u}_x}_{\text{Not necessarily} = 0!}$$

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$$x^2 \hat{u}_t + t\hat{u}_x = x^2 (k\tilde{u})_t + t(k\tilde{u})_x =$$

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- Let u^* and $\tilde{u}$ solutions of the considered PDE, then

$$x^2 \tilde{u}_t + t\tilde{u}_x = 0 \qquad \text{and} \qquad x^2 u_t^* + t\ u_x^* = 0.$$

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  For $\hat{u} := u^* + \tilde{u}$ holds

$$x^2 \hat{u}_t + t \hat{u}_x = x^2 (u^* + \tilde{u})_t + t(u^* + \tilde{u})_x =$$

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  For $\hat{u} := u^* + \tilde{u}$ holds

$$x^2 \hat{u}_t + t\hat{u}_x = x^2 (u^* + \tilde{u})_t + t(u^* + \tilde{u})_x = 0$$

  What applies to the sum of 3 solutions?

Linearity of PDEs: Examples

Check whether multiples of solutions of the PDE

$$x^2 u_t + t u_x = 0$$

are solutions. If yes, check whether sums of solutions are also solutions.

- Assume $x^2 \tilde{u}_t + t \tilde{u}_x = 0$
  For $\hat{u} := k \cdot \tilde{u}$ with $k \in \mathbb{R}$, $k \neq 0$ holds

$$x^2 \hat{u}_t + t \hat{u}_x = x^2 (k \tilde{u})_t + t (k \tilde{u})_x = 0$$

- Let u^* and $\tilde{u}$ solutions of the considered PDE, then

$$x^2 \tilde{u}_t + t \tilde{u}_x = 0 \qquad \text{and} \qquad x^2 u^*_t + t\ u^*_x = 0.$$

  For $\hat{u} := u^* + \tilde{u}$ holds

$$x^2 \hat{u}_t + t \hat{u}_x = x^2 (u^* + \tilde{u})_t + t (u^* + \tilde{u})_x = 0$$

  What applies to the sum of 3 solutions? Is also a solution
  What applies for any number of linear combinations of solutions?

Linearity of PDEs: Examples

Check whether multiples of solutions of the PDE

$$x^2 u_t + t u_x = 0$$

are solutions. If yes, check whether sums of solutions are also solutions.

- Assume $x^2 \tilde{u}_t + t \tilde{u}_x = 0$
  For $\hat{u} := k \cdot \tilde{u}$ with $k \in \mathbb{R}$, $k \neq 0$ holds

$$x^2 \hat{u}_t + t \hat{u}_x = x^2 (k\tilde{u})_t + t(k\tilde{u})_x = 0$$

- Let u^* and $\tilde{u}$ solutions of the considered PDE, then

$$x^2 \tilde{u}_t + t \tilde{u}_x = 0 \qquad \text{and} \qquad x^2 u^*_t + t \; u^*_x = 0.$$

  For $\hat{u} := u^* + \tilde{u}$ holds

$$x^2 \hat{u}_t + t \hat{u}_x = x^2 (u^* + \tilde{u})_t + t(u^* + \tilde{u})_x = 0$$

  What applies to the sum of 3 solutions? Is also a solution

  What applies for any number of linear combinations of solutions? Is also a solution

Superposition

Let f be a continuously differentiable function. Show that

$$u(t,x) = f\left(\frac{x^3}{3} - \frac{t^2}{2}\right)$$

is a solution of $x^2 u_t + t u_x = 0$.

Superposition

Let f be a continuously differentiable function. Show that

$$u(t,x) = f\left(\frac{x^3}{3} - \frac{t^2}{2}\right)$$

is a solution of $x^2 u_t + t u_x = 0$.

$$u_t(t,x) = \left(f\left(\frac{x^3}{3} - \frac{t^2}{2}\right)\right)_t =$$

$$u_x(t,x) = \left(f\left(\frac{x^3}{3} - \frac{t^2}{2}\right)\right)_x =$$

Superposition

Let f be a continuously differentiable function. Show that

$$u(t,x) = f\left(\frac{x^3}{3} - \frac{t^2}{2}\right)$$

is a solution of $x^2 u_t + t u_x = 0$.

chain rule!

$$u_t(t,x) = \left(f\left(\frac{x^3}{3} - \frac{t^2}{2}\right)\right)_t = (-t) f'\left(\frac{x^3}{3} - \frac{t^2}{2}\right),$$

chain rule!

$$u_x(t,x) = \left(f\left(\frac{x^3}{3} - \frac{t^2}{2}\right)\right)_x = x^2 f'\left(\frac{x^3}{3} - \frac{t^2}{2}\right).$$

Thus, $x^2 u_t + t u_x = 0$.

$$= x^2 (-t) f'\left(\frac{x^3}{3} - \frac{t^2}{2}\right) + t x^2 f'\left(\frac{x^3}{3} - \frac{t^2}{2}\right)$$

$$= (-t + t) x^2 f'\left(\frac{x^3}{3} - \frac{t^2}{2}\right) = 0$$

Superposition

Show that $\tilde{u}(t,x) = x^6 - 3t^2x^3 + \frac{9}{4}t^4$ is a solution of the IVP

$$\begin{cases} x^2 u_t + t u_x = 0 & \text{for} \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = x^6 & \text{for} \quad t = 0,\ x \in \mathbb{R}, \end{cases}$$

Show that $\hat{u}(t,x) = \cos\left(\frac{x^3}{3} - \frac{t^2}{2}\right)$ is a solution of the IVP

$$\begin{cases} x^2 u_t + t u_x = 0 & \text{for} \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = \cos\left(\frac{x^3}{3}\right) & \text{for} \quad t = 0,\ x \in \mathbb{R}. \end{cases}$$

Find a solution to

$$\begin{cases} x^2 u_t + t u_x = 0, & \text{for} \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = 4x^6 - 2\cos\left(\frac{x^3}{3}\right) & \text{for} \quad t = 0,\ x \in \mathbb{R}, \end{cases}$$

Superposition

Show that $\tilde{u}(t,x) = x^6 - 3t^2x^3 + \frac{9}{4}t^4$ is a solution of the IVP

$$\begin{cases} x^2 u_t + t u_x = 0 & \text{for} \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = x^6 & \text{for} \quad t = 0,\ x \in \mathbb{R}, \end{cases}$$

Straightforward or $\tilde{u}(t,x) = \left(x^3 - \frac{3}{2}t^2\right)^2 = f\left(\frac{x^3}{3} - \frac{t^2}{2}\right)$ with $f(y) = 9y^2$ use → Slide (*)
↳ Insert

Show that $\hat{u}(t,x) = \cos\left(\frac{x^3}{3} - \frac{t^2}{2}\right)$ is a solution of the IVP

$$\begin{cases} x^2 u_t + t u_x = 0 & \text{for} \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = \cos\left(\frac{x^3}{3}\right) & \text{for} \quad t = 0,\ x \in \mathbb{R}. \end{cases}$$

Straightforward or Slide (*) with $f = \cos$
↳ Insert

Find a solution to

$$\begin{cases} x^2 u_t + t u_x = 0, & \text{for} \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = 4x^6 - 2\cos\left(\frac{x^3}{3}\right) & \text{for} \quad t = 0,\ x \in \mathbb{R}, \end{cases}$$

Slide (*)

Linear combinations of solutions to the PDE are solutions (shown before), hence, $4\tilde{u} - 2\hat{u}$ solves this IVP. → $4\tilde{u}(0,x) - 2\hat{u}(0,x) = 4x^6 - 2\cos\left(\frac{x^3}{3}\right)$

Linear substitution

Solve the initial value problem

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0 \quad \text{for } x \in \mathbb{R},\ t \in \mathbb{R}^+$$
$$u(x,0) = 0 \quad \text{for } x \in \mathbb{R},$$
$$u_t(x,0) = 2xe^{-x^2} \quad \text{for } x \in \mathbb{R}.$$

Hint: Use the substitution $\alpha = x + \frac{t}{4}$, $\mu = x - t$ to transform the PDE

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Hint: Use the substitution $\alpha = x + \frac{t}{4}$, $\mu = x - t$ to transform the PDE

Define $v(\alpha,\mu) := u(x(\alpha,\mu), t(\alpha,\mu))$ with $\begin{cases} x(\alpha,\mu) = \frac{4\alpha+\mu}{5} \\ t(\alpha,\mu) = \frac{4\alpha-4\mu}{5} \end{cases}$.

Then use the chain rule:

$$v_\alpha = u_x \cdot \frac{dx}{d\alpha} + u_t \cdot \frac{dt}{d\alpha} =$$

$$v_{\alpha\mu} = \left(u_{xx} \cdot \frac{dx}{d\alpha} \cdot \frac{dx}{d\mu} + u_{xt} \cdot \frac{dx}{d\alpha} \cdot \frac{dt}{d\mu}\right) + \left(u_{tx} \cdot \frac{dt}{d\alpha} \cdot \frac{dx}{d\mu} + u_{tt} \cdot \frac{dt}{d\alpha} \cdot \frac{dt}{d\mu}\right)$$

$$=$$

$$\alpha = x + \frac{t}{4}, \quad \mu = x - t, \quad \alpha - \mu = \frac{5}{4}t \iff t = \frac{4\alpha - 4\mu}{5} \Rightarrow \mu = x - \frac{4\alpha-4\mu}{5} \iff x = \frac{4\alpha+\mu}{5}$$

$$\frac{d}{d\alpha}\frac{4\alpha+\mu}{5} = \frac{4}{5}, \quad \frac{d}{d\alpha}\frac{4\alpha-4\mu}{5} = \frac{4}{5}, \quad \frac{d}{d\mu}\frac{4\alpha+\mu}{5} = \frac{1}{5}, \quad \frac{d}{d\mu}\frac{4\alpha-4\mu}{5} = -\frac{4}{5}$$

Linear substitution

Solve the initial value problem

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0 \quad \text{for } x \in \mathbb{R},\ t \in \mathbb{R}^+$$
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Hint: Use the substitution $\alpha = x + \frac{t}{4}$, $\mu = x - t$ to transform the PDE

Define $v(\alpha,\mu) := u(x(\alpha,\mu), t(\alpha,\mu))$ with $\begin{cases} x(\alpha,\mu) = \frac{4\alpha+\mu}{5} \\ t(\alpha,\mu) = \frac{4\alpha-4\mu}{5} \end{cases}$.

Then use the chain rule:

$$v_\alpha = \frac{4}{5}u_x + \frac{4}{5}u_t$$
$$v_{\alpha\mu} = \frac{4}{25}\left(u_{xx} - 4u_{xt} + u_{xt} - 4u_{tt}\right)$$

Linear substitution

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0 \quad \text{for } x \in \mathbb{R},\ t \in \mathbb{R}^+$$

$$u(x,0) = 0, \qquad u_t(x,0) = 2xe^{-x^2} \quad \text{for } x \in \mathbb{R}.$$

$$v(\alpha,\mu) := u(x(\alpha,\mu), t(\alpha,\mu)) \text{ with } \begin{cases} x(\alpha,\mu) = \frac{4\alpha+\mu}{5} \\ t(\alpha,\mu) = \frac{4\alpha-4\mu}{5} \end{cases}.$$

$$v_{\alpha\mu} = \frac{4}{25}\left(u_{xx} - 4u_{xt} + u_{xt} - 4u_{tt}\right)$$

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$$v_{\alpha\mu} = \frac{4}{25}\left(u_{xx} - 4u_{xt} + u_{xt} - 4u_{tt}\right)$$

Thus, a C^2 function $u(x,t)$ is a solution of the considered PDE if and only if v solves

$$\boxed{v_{\alpha\mu} = 0.}$$

Linear substitution

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Thus, a C^2 function $u(x,t)$ is a solution of the considered PDE if and only if v solves

$$\boxed{v_{\alpha\mu} = 0.}$$

In particular, v_α is independent of μ, i.e.,

$$v(\alpha,\mu)_\alpha = \phi(\alpha) \overset{\int d\alpha}{\Longrightarrow} v(\alpha,\mu) =$$

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In particular, v_α is independent of μ, i.e.,

$$v(\alpha,\mu)_\alpha = \phi(\alpha) \overset{\int d\alpha}{\Longrightarrow} v(\alpha,\mu) = \Phi(\alpha) + \Psi(\mu)$$

Linear substitution

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$$v_{\alpha\mu} = \frac{4}{25}\left(u_{xx} - 4u_{xt} + u_{xt} - 4u_{tt}\right)$$

Thus, a C^2 function $u(x,t)$ is a solution of the considered PDE if and only if v solves

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In particular, v_α is independent of μ, i.e.,

$$v(\alpha,\mu)_\alpha = \phi(\alpha) \overset{\int d\alpha}{\Longrightarrow} v(\alpha,\mu) = \Phi(\alpha) + \Psi(\mu)$$

Hence, every function of the form

$$u(x,t) = \Phi(x + \frac{t}{4}) + \Psi(x - t)$$

is a solution to the original PDE (if Φ, Ψ 'smooth enough').

Linear substitution

$v(\alpha, \mu) := u(x(\alpha, \mu), t(\alpha, \mu))$ with $\begin{cases} x(\alpha, \mu) = \frac{4\alpha+\mu}{5} \\ t(\alpha, \mu) = \frac{4\alpha-4\mu}{5} \end{cases}$.

Every function of the form $u(x, t) = \Phi(x + \frac{t}{4}) + \Psi(x - t)$ is a solution to the original PDE. (Φ, Ψ 'smooth enough')

Now consider the initial values:

$$u(x, t) = v(\alpha, \mu) = \Phi(x+t/4)+\Psi(x-t) \Rightarrow u_t(x, t) =$$

Linear substitution

$v(\alpha, \mu) := u(x(\alpha, \mu), t(\alpha, \mu))$ with $\begin{cases} x(\alpha, \mu) = \frac{4\alpha+\mu}{5} \\ t(\alpha, \mu) = \frac{4\alpha-4\mu}{5} \end{cases}$.

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Now consider the initial values:

chainrule

$$u(x, t) = v(\alpha, \mu) = \Phi(x+t/4)+\Psi(x-t) \Rightarrow u_t(x, t) = \frac{1}{4}\Phi'(x+t/4)-\Psi'(x-t)$$

Linear substitution

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Every function of the form $u(x, t) = \Phi(x + \frac{t}{4}) + \Psi(x - t)$ is a solution to the original PDE. (Φ, Ψ 'smooth enough')

Now consider the initial values:

$$u(x, t) = v(\alpha, \mu) = \Phi(x+t/4)+\Psi(x-t) \Rightarrow u_t(x, t) = \frac{1}{4}\Phi'(x+t/4)-\Psi'(x-t)$$

$$u(x, 0) =$$

Linear substitution

$v(\alpha, \mu) := u(x(\alpha, \mu), t(\alpha, \mu))$ with $\begin{cases} x(\alpha, \mu) = \frac{4\alpha+\mu}{5} \\ t(\alpha, \mu) = \frac{4\alpha-4\mu}{5} \end{cases}$.

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$$u_t(x,0) \overset{!}{=} 2xe^{-x^2} \Longrightarrow$$

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Integrate

$$u_t(x,0) \overset{!}{=} 2xe^{-x^2} \Longrightarrow \frac{1}{4}\Phi(x) - \Psi(x) \overset{!}{=} \int_{x_0}^{x} 2ze^{-z^2}\,dz = -e^{-x^2} + c$$

Linear substitution

$v(\alpha,\mu) := u(x(\alpha,\mu), t(\alpha,\mu))$ with $\begin{cases} x(\alpha,\mu) = \frac{4\alpha+\mu}{5} \\ t(\alpha,\mu) = \frac{4\alpha-4\mu}{5} \end{cases}$.

Every function of the form $u(x,t) = \Phi(x+\frac{t}{4}) + \Psi(x-t)$ is a solution to the original PDE. (Φ, Ψ 'smooth enough')

Now consider the initial values:

$$u(x,t) = v(\alpha,\mu) \overset{(*)}{=} \Phi(\underbrace{x+t/4}_{=\alpha}) + \Psi(\underbrace{x-t}_{=\mu}) \Rightarrow u_t(x,t) = \frac{1}{4}\Phi'(x+t/4) - \Psi'(x-t)$$

chain rule

$$u(x,0) = \Phi(x) + \Psi(x) \qquad u_t(x,0) = \tfrac{1}{4}\Phi'(x)$$

$$u(x,0) \overset{!}{=} 0 \qquad \Longrightarrow \Phi(x) \overset{(*)}{=} -\Psi(x)$$

$$u_t(x,0) \overset{!}{=} 2xe^{-x^2} \Longrightarrow \frac{1}{4}\Phi(x) - \Psi(x) \overset{!}{=} \int_{x_0}^{x} 2ze^{-z^2}\,dz = -e^{-x^2} + c$$

$$\tfrac{1}{4}\Phi(x) - \underbrace{\Psi(x)}_{-\Phi(x)\ (*)} = -e^{-x^2} + c$$

$$\Leftrightarrow \tfrac{5}{4}\Phi(x) = -e^{-x^2} + c, \quad c \in \mathbb{R}$$

$$\Leftrightarrow \Phi(x) = -\tfrac{4}{5}e^{-x^2} + c \overset{(*)}{=} -\Psi(x)$$

Hence, the solution of the initial value problem is given by

$$u(x,t) \overset{(*)}{=} \Phi\left(x + \frac{t}{4}\right) + \Psi(x-t) = -\frac{4}{5}e^{-(x+\frac{t}{4})^2} + \frac{4}{5}e^{-(x-t)^2}.$$

Remark: The substitution of $u_{tt} + (a+b)u_{tx} + abu_{xx}$, the analogous substitution is given by $\alpha = x - bt$ $\mu = x - at$.